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# Bounds of minimal dilatation for pseudo-Anosovs and the magic 3-manifold

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## 1 Minimal dilatation of pseudo-Anosovs

Let  $\Sigma = \Sigma_{g,n}$  be an orientable surface of genus  $g$  with  $n$  punctures, and let  $\text{Mod}(\Sigma)$  be the mapping class group. Mapping classes  $\phi \in \text{Mod}(\Sigma)$  are classified into 3 types, periodic, reducible, pseudo-Anosov. There exist two numerical invariants of pseudo-Anosov mapping classes. One is the entropy  $\text{ent}(\phi)$  which is the logarithm of the dilatation  $\lambda(\phi) > 1$ . The other is the volume  $\text{vol}(\phi)$  which is the hyperbolic volume of the mapping torus of  $\phi$

$$\mathbb{T}(\phi) = \Sigma \times [0, 1] / \sim,$$

where  $\sim$  identifies  $(x, 1)$  with  $(f(x), 0)$  for any representative  $f \in \phi$ .

We denote by  $\delta_{g,n}$ , the minimal dilatation for pseudo-Anosov elements  $\phi \in \text{Mod}(\Sigma_{g,n})$ . We set  $\delta_g = \delta_{g,0}$ . A natural question arises.

**Question 1.1.** *What is the value of  $\delta_{g,n}$ ? Find a pseudo-Anosov element of  $\text{Mod}(\Sigma_{g,n})$  whose dilatation is equal to  $\delta_{g,n}$ .*

The above question is hard in general. For instance, in the case of closed surfaces, it is open to determine the values  $\delta_g$  for  $g \geq 3$ . On the other hand, one understands the asymptotic behavior of the minimal entropy  $\log \delta_g$ . Penner proved that  $\log \delta_g \asymp \frac{1}{g}$  [12]. The following question posed by McMullen.

**Question 1.2** ([11]). *Does  $\lim_{g \rightarrow \infty} g \log \delta_g$  exist? What is its value?*

Note that  $\lim_{g \rightarrow \infty} g \log \delta_g$  exists if and only if  $\lim_{g \rightarrow \infty} |\chi(\Sigma_g)| \log \delta_g$  exists, where  $\chi(\Sigma)$  is the Euler characteristic of  $\Sigma$ .

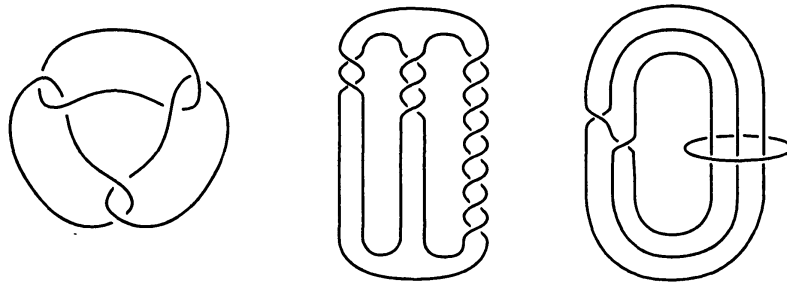


Figure 1: (left) 3 chain link  $\mathcal{C}_3$ . (center)  $(-2, 3, 8)$ -pretzel link or Whitehead sister link. (right) link  $6_2^2$ . ( $N$  equals the exterior of  $\mathcal{C}_3$ .  $N(\frac{-3}{2})$  is homeomorphic to the  $(-2, 3, 8)$ -pretzel link exterior.  $N(\frac{-1}{2})$  is homeomorphic to the  $6_2^2$  link exterior.)

Related questions on the minimal dilatation are ones for orientable pseudo-Anosovs. A pseudo-Anosov mapping class  $\phi$  is said to be *orientable* if the invariant (un)stable foliation for a pseudo-Anosov homeomorphism  $\Phi \in \phi$  is orientable. We denote by  $\delta_g^+$ , the minimal dilatation for orientable pseudo-Anosov elements of  $\text{Mod}(\Sigma_g)$  for a closed surface  $\Sigma_g$  of genus  $g$ .

In this paper, we report our results in [7, 8] on the minimal dilatation by investigating the so called *magic manifold*  $N$  which is the exterior of the 3 chain link  $\mathcal{C}_3$ , see Figure 1(left). In Section 2, we describe a motivation for the study of pseudo-Anosovs which occur as the monodromies on fibers for Dehn fillings of  $N$ . In Section 3, we state our results.

We would like to note that this paper only contains some results in [7, 8] and does not contain their proofs. The readers who are interested in the details should consult (the introduction of) [7, 8].

## 2 Why is the magic manifold an intriguing example?

Gordon and Wu named the exterior of the link  $\mathcal{C}_3$  the *magic manifold*  $N$ , see [3]. The reason why this manifold is called “magic” is that many important examples for the study of the exceptional Dehn fillings can be obtained from the Dehn fillings of a single manifold  $N$ . The magic

manifold is fibered and it has the smallest known volume among orientable hyperbolic 3-manifolds having 3 cusps. Many manifolds having at most 2 cusps with small volume are obtained from  $N$  by Dehn fillings, see [10].

## 2.1 Entropy versus volume

Both invariants entropy  $\text{ent}(\phi)$  and volume  $\text{vol}(\phi)$  know some complexity of pseudo-Anosovs  $\phi$ . A natural question is how these are related.

**Theorem 2.1** ([6]). *There exists a constant  $B = B(\Sigma)$  depending only on the topology of  $\Sigma$  such that the inequality,*

$$B \text{vol}(\phi) \leq \text{ent}(\phi)$$

*holds for any pseudo-Anosov  $\phi$  on  $\Sigma$ . Furthermore, for any  $\varepsilon > 0$ , there exists a constant  $C = C(\varepsilon, \Sigma) > 1$  depending only on  $\varepsilon$  and the topology of  $\Sigma$  such that the inequality*

$$\text{ent}(\phi) \leq C \text{vol}(\phi)$$

*holds for any pseudo-Anosov  $\phi$  on  $\Sigma$  whose mapping torus  $\mathbb{T}(\phi)$  has no closed geodesics of length  $< \varepsilon$ .*

The first part of Theorem 2.1 says that if the entropy is small, the volume can not be large.

For a non-negative integer  $c$ , we set

$$\begin{aligned} \lambda(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \text{Mod}(\Sigma), \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}, \\ \text{vol}(\Sigma; c) &= \min\{\text{vol}(\phi) \mid \phi \in \text{Mod}(\Sigma), \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}. \end{aligned}$$

A variation on the questions of the minimal dilatations is to determine  $\lambda(\Sigma; c)$  and to find a mapping class realizing the minimum. In [6], the authors and S. Kojima obtain experimental results concerning the minimal dilatation. In the case the mapping class group  $\text{Mod}(D_n)$  of an  $n$ -punctured disk  $D_n$ , they observe that for many pairs  $(n, c)$ , there exists a pseudo-Anosov element simultaneously reaching both minima  $\lambda(D_n; c)$  and  $\text{vol}(D_n; c)$ . Experiments tell us that in case  $c = 3$ , the mapping tori reaching both minima are homeomorphic to  $N$ . Moreover when  $c = 2$ , it is observed that there exists a mapping class  $\phi$

realizing both  $\lambda(D_n; 2)$  and  $\text{vol}(D_n; 2)$  and its mapping torus  $\mathbb{T}(\phi)$  is homeomorphic to a Dehn filling of  $N$  along one cusp. This is a motivation for us for focusing on  $N$ .

## 2.2 Small dilatation pseudo-Anosovs

After we have finished our papers [6, 7], we learned the small dilatation pseudo-Anosovs, introduced by Farb, Leininger and Margalit.

For any number  $P > 1$ , define the set of pseudo-Anosov homeomorphisms

$$\Psi_P = \{\text{pseudo-Anosov } \Phi : \Sigma \rightarrow \Sigma \mid \chi(\Sigma) < 0, |\chi(\Sigma)| \log \lambda(\Phi) \leq \log P\}.$$

Elements  $\Phi \in \Psi_P$  are called *small dilatation pseudo-Anosov homeomorphisms*. If one takes  $P$  sufficiently large (e.g.  $P \geq 2 + \sqrt{3}$ ), then  $\Psi_P$  contains a pseudo-Anosov homeomorphism  $\Phi_g : \Sigma_g \rightarrow \Sigma_g$  for each  $g \geq 2$ . By a result in [5],  $\Psi_P$  also contains pseudo-Anosov homeomorphism  $\Phi_n : D_n \rightarrow D_n$  for each  $n \geq 3$ . Let  $\Sigma^\circ \subset \Sigma$  be the surface obtained by removing the singularities of the (un)stable foliation for  $\Phi$  and  $\Phi|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \Sigma^\circ$  denotes the restriction. Observe that  $\lambda(\Phi) = \lambda(\Phi|_{\Sigma^\circ})$ . The set

$$\Psi_P^\circ = \{\Phi|_{\Sigma^\circ} : \Sigma^\circ \rightarrow \Sigma^\circ \mid (\Phi : \Sigma \rightarrow \Sigma) \in \Psi_P\}$$

is infinite. Let  $\mathcal{T}(\Psi_P^\circ)$  be the set of homeomorphism classes of mapping tori by elements of  $\Psi_P^\circ$ .

**Theorem 2.2** ([2]). *The set  $\mathcal{T}(\Psi_P^\circ)$  is finite. Namely, for each  $P > 1$ , there exist finite many complete, non compact hyperbolic 3-manifolds  $M_1, M_2, \dots, M_r$  fibering over  $S^1$  so that the following holds. Any pseudo-Anosov  $\Phi \in \Psi_P$  occurs as the monodromy of a Dehn filling of one of the  $M_k$ . In particular, there exists a constant  $V = V(P)$  such that  $\text{vol}(\Phi) \leq V$  holds for any  $\Phi \in \Psi_P$ .*

It is not known that how large the set of manifolds  $\{M_1, \dots, M_r\}$  is. By Theorem 2.2, one sees that the following set  $\mathcal{V}$  is finite.

$$\mathcal{V} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid n \geq 3, \Phi \text{ is pseudo-Anosov on } \Sigma = D_n, \lambda(\Phi) = \delta(D_n)\},$$

where  $\delta(D_n)$  denotes the minimal dilatation for pseudo-Anosov elements of  $\text{Mod}(D_n)$  on  $D_n$ .

In [7], we show that for each  $n \geq 9$  (resp.  $n = 3, 4, 5, 7, 8$ ), there exists a pseudo-Anosov homeomorphism  $\Phi_n : D_n \rightarrow D_n$  with the smallest known entropy (resp. the smallest entropy) which occurs as the monodromy on an  $n$ -punctured disk fiber for the Dehn filling of  $N$ . A pseudo-Anosov homeomorphism  $\Phi_6 : D_6 \rightarrow D_6$  with the smallest entropy occurs as the monodromy on a 6-punctured disk fiber for  $N$ . In particular,  $N \in \mathcal{V}$ . See also work of Venzke [13]. This result suggests that one may have a chance to find pseudo-Anosov homeomorphisms with small dilatation on other surfaces which arise as the monodromies on fibers for Dehn fillings of  $N$ . This is another motivation for us.

### 3 Results

Let us introduce the following polynomial

$$f_{(k,\ell)}(t) = t^{2k} - t^{k+\ell} - t^k - t^{k-\ell} + 1 \text{ for } k > 0, -k < \ell < k,$$

having a unique real root  $\lambda_{(k,\ell)}$  greater than 1 [8]. For the rational number  $r$ , let  $N(r)$  be the Dehn filling of  $N$  along the slope  $r$ .

**Theorem 3.1.** *Let  $r \in \{-\frac{3}{2}, -\frac{1}{2}, 2\}$ . For each  $g \geq 3$ , there exists a monodromy  $\Phi_g = \Phi_g(r)$  on a closed fiber of genus  $g$  for a Dehn filling of  $N(r)$ , where the filling is on the boundary slope of a fiber of  $N(r)$ , such that*

$$\lim_{g \rightarrow \infty} g \log \lambda(\Phi_g) = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

*In particular*

$$\lim_{g \rightarrow \infty} \sup g \log \delta_g \leq \log\left(\frac{3+\sqrt{5}}{2}\right).$$

**Remark 3.2.** *Independently, Hironaka has obtained Theorem 3.1 in case  $r = -\frac{1}{2}$  [4]. Independently, Aaber and Dunfield have obtained Theorem 3.1 in case  $r = -\frac{3}{2}$  [1]. They have obtained similar results on the dilatation to those given in [8].*

By using monodromies on closed fibers coming from  $N(-\frac{3}{2})$ , we find an upper bound of  $\delta_g$  for  $g \equiv 0, 1, 5, 6, 7, 9 \pmod{10}$  and  $g \geq 5$ .

**Theorem 3.3.** (1)  $\delta_g \leq \lambda_{(g+2,1)}$  if  $g \equiv 0, 1, 5, 6 \pmod{10}$  and  $g \geq 5$ .

(2)  $\delta_g \leq \lambda_{(g+2,2)}$  if  $g \equiv 7, 9 \pmod{10}$  and  $g \geq 7$ .

For more details of an upper bound of  $\delta_g$  for other  $g$  (e.g.  $g \equiv 2, 4 \pmod{10}$ ), see [8]. The bound in Theorem 3.3 improves the one by Hironaka [4].

We turn to the study on the minimal dilatations  $\delta_g^+$  for orientable pseudo-Anosovs. The minima  $\delta_g^+$  were determined for  $g = 2$  by Zhurov [14], for  $3 \leq g \leq 5$  by Lanneau-Thiffeault [9], and for  $g = 8$  by Lanneau-Thiffeault and Hironaka [9, 4]. Those values are given by  $\delta_2^+ = \lambda_{(2,1)}$ ,  $\delta_3^+ = \lambda_{(3,1)} = \lambda_{(4,3)} \approx 1.40127$ ,  $\delta_4^+ = \lambda_{(4,1)} \approx 1.28064$ ,  $\delta_5^+ = \lambda_{(6,1)} = \lambda_{(7,4)} \approx 1.17628$  and  $\delta_8^+ = \lambda_{(8,1)} \approx 1.12876$ .

We recall the lower bounds of  $\delta_6^+$  and  $\delta_7^+$  and the question on  $\delta_g^+$  for  $g$  even by Lanneau-Thiffeault.

**Theorem 3.4** ([9]).

(1)  $\delta_6^+ \geq \lambda_{(6,1)} \approx 1.17628$ .

(2)  $\delta_7^+ \geq \lambda_{(9,2)} \approx 1.11548$ .

**Question 3.5** ([9]). *For  $g$  even, is  $\delta_g^+$  equal to  $\lambda_{(g,1)}$ ?*

We give an upper bound of  $\delta_g^+$  in case  $g \equiv 1, 5, 7, 9 \pmod{10}$  using orientable pseudo-Anosov monodromies coming from  $N(\frac{-3}{2})$ .

**Theorem 3.6.** (1)  $\delta_g^+ \leq \lambda_{(g+2,2)}$  if  $g \equiv 7, 9 \pmod{10}$  and  $g \geq 7$ .

(2)  $\delta_g^+ \leq \lambda_{(g+2,4)}$  if  $g \equiv 1, 5 \pmod{10}$  and  $g \geq 5$ .

The bound in Theorem 3.6 improves the one by Hironaka [4]. Theorem 3.6(1) together with Theorem 3.4(2) gives:

**Corollary 3.7.**  $\delta_7^+ = \lambda_{(9,2)}$ .

Independently, Corollary 3.7 was established by Aaber and Dunfield [1].

The following tells us that the sequence  $\{\delta_g^+\}_{g \geq 2}$  is not monotone decreasing.

**Proposition 3.8.** *If Question 3.5 is true, then  $\delta_g^+ < \delta_{g+1}^+$  whenever  $g \equiv 1, 5, 7, 9 \pmod{10}$  and  $g \geq 7$ . In particular the inequality  $\delta_7^+ < \delta_8^+$  holds.*

Our pseudo-Anosov homeomorphisms providing the upper bound of  $\delta_g$  in Theorem 3.3(1) are not orientable. This together with the inequality  $\lambda_{(7,1)} < \lambda_{(6,1)} = \delta_5^+$  implies:

**Corollary 3.9.**  $\delta_5 < \delta_5^+$ .

## References

- [1] J. W. Aaber and N. M. Dunfield, *Closed surface bundles of least volume*, preprint, arXiv:1002.3423
- [2] B. Farb, C. J. Leininger and D. Margalit, *Small dilatation pseudo-Anosovs and 3-manifolds*, preprint, arXiv:0905.0219
- [3] C. Gordon and Y-Q. Wu, *Toroidal and annular Dehn fillings*, Proceedings of the London Mathematical Society (3) 78 (1999), 662-700.
- [4] E. Hironaka, *Small dilatation pseudo-Anosov mapping classes coming from the simplest hyperbolic braid*, preprint, arXiv:0909.4517
- [5] E. Hironaka and E. Kin, *A family of pseudo-Anosov braids with small dilatation*, Algebraic and Geometric Topology 6 (2006), 699-738.
- [6] E. Kin, S. Kojima and M. Takasawa, *Entropy versus volume for pseudo-Anosovs*, Experimental Mathematics 18 (2009), 397-407.
- [7] E. Kin and M. Takasawa, *Pseudo-Anosov braids with small entropy and the magic 3-manifold*, preprint, arXiv:0812.4589
- [8] E. Kin and M. Takasawa, *Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior*, preprint, arXiv:1003.0545
- [9] E. Lanneau and J. L. Thiffeault, *On the minimum dilatation of pseudo-Anosov homeomorphisms on surfaces of small genus*, Annales de l'Institut Fourier, in press.



- [10] B. Martelli and C. Petronio, *Dehn filling of the “magic” 3-manifold*, Communications in Analysis and Geometry 14 (2006), 969-1026.
- [11] C. McMullen, *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesic for foliations*, Annales Scientifiques de l’École Normale Supérieure. Quatrième Série 33 (2000), 519-560.
- [12] R. C. Penner, *Bounds on least dilatations*, Proceedings of the American Mathematical Society 113 (1991) 443-450.
- [13] R. Venzke, *Braid forcing, hyperbolic geometry, and pseudo-Anosov sequences of low entropy*, PhD thesis, California Institute of Technology (2008), available at <http://etd.caltech.edu/etd/available/etd-05292008-085545/>
- [14] A. Y. Zhironov, *On the minimum dilation of pseudo-Anosov diffeomorphisms on a double torus*, Russian Mathematical Surveys 50 (1995), 223-224.

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